

DOUBLY PERIODIC WAVE SOLUTIONS OF HIGHER ORDER KdV-mKdV EQUATIONS

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Abstract

The Weierstrass elliptic function method has been employed to derive the periodic wave solutions and the corresponding explode decay mode solutions or solitary wave solutions of the generalized fifth-order Kortweg-de Vries equation, a modified fifth order Korteweg-de Vries equation and a coupled Schrodinger-Korteweg de Vries system.

1. Introduction

Nonlinear evolution equations (NLEEs) [1] are widely used to describe various physical phenomena and investigation of cnoidal wave solutions and their infinite period counterparts, namely, explode decay mode solutions (EDMSs) or solitary wave solutions (SWSs) have been a hot 2000 Mathematics Subject Classification: 02.30 Hq, 02.30 lk and 02.30 Jr.

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topic of research for decades. The physical significance of the solutions of these equations are that they describe various natural phenomena such as vibrations, solitons etc. The salient feature of these equations is the study of the travelling wave solutions (TWSs) which are solutions of constant form moving with a fixed velocity. Three important types of travelling waves are: the solitary waves which are localized travelling waves asymptotically zero at large distances, the periodic waves, and the kink-antikink waves which rise or descend from one asymptotic state to another. The most popular methods for deriving these exact solutions of NLEEs are, for example, Backlund transformation [17], inverse scattering technique [3] Hirota's direct method [10], tanh method [19], series method [9, 13], Jacobi elliptic function method and its extensions [6, 18], the algebraic method [4], the sinh-Gordon equation expansion method [23, 16], and a mapping method and its extensions [20-22].

In this paper, we make use of the properties of the Weierstrass elliptic functions (WEFs) [2, 14, 15, 12] to derive the periodic wave solutions (PWSs) in terms of Jacobi elliptic functions (JEFs) of some NLEEs and, then derive the EDMSs or the SWSs which are their infinite period counterparts.

We organize our work in the following way: In Section 2, we give an account of the representation of the solutions of higher order ordinary differential equations (ODEs) in terms of WEFs and of the relation between the WEFs and JEFs. In Section 3, we derive both the elliptic function solutions and the corresponding EDMSs and SWSs of a generalized fifth-order Korteweg- de Vries (KdV) equation [11]. In Section 4, we investigate the PWSs, EDMSs and SWSs of a fifth order modified Korteweg-de Vries equation (mKdV) [5]. In Section 5, we derive the PWSs and the EDMSs of a Schrodinger-KdV system [3]. In Section 6, we conclude the work.

2. Weierstrass Elliptic Function

We consider the ODE of order $2p$ given by

$$\frac{d^{2p}\phi}{d\xi^{2p}} = f(\phi; r + 1), \quad (2.1)$$

where $f(\phi; r + 1)$ is an $(r + 1)$ degree polynomial in ϕ .

We assume that

$$\phi = A + B\wp^{(2s)}(\xi) \quad (2.2)$$

is a solution of Equation (2.1), where A and B are arbitrary constants and $\wp^{(2s)}(\xi)$ is the $(2s)$ -th derivative of the WEF $\wp(\xi)$. It can be shown that the $(2s)$ -th derivative of $\wp(\xi)$ is an $(s + 1)$ degree polynomial in $\wp(\xi)$ itself.

Therefore, for (2.2) to be a solution of Equation (2.1), we should have the relation

$$p - r = rs. \quad (2.3)$$

So, it is necessary that $p \geq r$ for us to assume a solution in the form (2.2). But this is in no way a sufficient condition for the existence of the periodic wave solution in the form (2.2).

WEF $\wp(\xi; g_2, g_3)$ with invariants g_2 and g_3 , satisfy

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \quad (2.4)$$

where

$$g_2^3 - 27g_3^2 > 0. \quad (2.5)$$

The WEF $\wp(\xi; g_2, g_3)$ is related to the JEFs $ns(\xi; m)$, $ds(\xi; m)$ and $cs(\xi; m)$, which are equivalent expressions given by

$$\wp(\xi; g_2, g_3) = e_3 + (e_1 - e_3)ns^2(\xi), \quad (2.6a)$$

$$\wp(\xi; g_2, g_3) = e_2 + (e_1 - e_3)ds^2(\xi), \quad (2.6b)$$

$$\wp(\xi; g_2, g_3) = e_1 + (e_1 - e_3)cs^2(\xi), \quad (2.6c)$$

where e_1, e_2, e_3 satisfy

$$4z^3 - g_2z - g_3 = 0, \quad (2.7)$$

with

$$e_1 > e_2 > e_3, \quad (2.8)$$

and

$$m^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad (2.9)$$

where m is the modulus of the JEF.

In this case, the infinite period counterpart will give rise to only EDMSSs.

To get the SWSs, we will have to consider solutions in terms of reciprocal WEFs. For this purpose, we consider the ODE of order $2k$ given by

$$\frac{d^{2k}\phi}{d\xi^{2p}} = f(\phi; r + 1), \quad (2.10)$$

where $f(\phi; r + 1)$ is an $(r + 1)$ degree polynomial in ϕ .

We assume that

$$\phi = A + BQ^{(2s)}(\xi) \quad (2.11)$$

is a solution of Equation (2.10), where A and B are arbitrary constants and $Q^{(2s)}(\xi)$ is the $(2s)$ -th derivative of the reciprocal WEF $Q(\xi) = \frac{1}{\wp(\xi)}$, $\wp(\xi)$ being the WEF. It can be shown that the $(2s)$ -th derivative of $Q(\xi)$ is a $(2s + 1)$ degree polynomial in $Q(\xi)$ itself.

Therefore, for (2.11) to be a solution of Equation (2.10), we should have the relation

$$2k - r = 2rs. \quad (2.12)$$

So, it is necessary that $2k \geq r$ for us to assume a solution in the form (2.11). But this is again in no way a sufficient condition for the existence of the periodic wave solution in the form (2.11).

3. Generalized Fifth Order KdV Equation

Consider the generalized fifth order KdV equation

$$u_t + \alpha u_{xxxxx} + \beta(uu_{xxx} + u_x u_{xx}) + \delta u^2 u_x = 0. \quad (3.1)$$

By considering the transformation

$$u(x, t) = u(z), z = x - wt, \quad (3.2)$$

the Equation (3.1) becomes

$$-wu' + \alpha u^{(5)} + \beta(uu'' + u'u'') + \delta u^2 u' = 0. \quad (3.3)$$

Integrating Equation (3.3) with respect to z , we get

$$-wu + \alpha u^{(4)} + \beta uu'' + \frac{\delta}{3}u^3 + K = 0, \quad (3.4)$$

where K is the integration constant.

We assume the solution of Equation (3.4) in the form

$$u = A + B\wp^{(2s)}(z), \quad (3.5)$$

where A and B are constants to be determined.

Balancing the highest derivative term and the nonlinear term, we will get $s = 0$ and so we can write the solution of Equation (3.4) as

$$u = A + B\wp(z). \quad (3.6)$$

Substituting Equation (3.5) into Equation (3.4) and equating the like powers of $wp(z)$, we arrive at

$$A = 0, \quad (3.7a)$$

$$B = \frac{-9\beta \pm 3\sqrt{9\beta^2 - 40\alpha\delta}}{\delta}, \quad (3.7b)$$

$$g_2 = \frac{-2w\delta}{36\alpha\delta - 9\beta^2 \pm 3\beta\sqrt{9\beta^2 - 40\alpha\delta}}, \quad (3.7c)$$

$$g_3 = \frac{K\delta}{\pm 36\alpha\sqrt{9\beta^2 - 40\alpha\delta} - 108\alpha\beta}. \quad (3.7d)$$

Thus our solution to Equation (3.4) is

$$u(z) = \frac{-9\beta \pm 3\sqrt{9\beta^2 - 40\alpha\delta}}{\delta}, \quad (3.8)$$

which can be written in terms of JEF as

$$u(z) = \frac{-9\beta \pm 3\sqrt{9\beta^2 - 40\alpha\delta}}{\delta} [e_3 + (e_1 - e_3)ns^2(z)]. \quad (3.9)$$

In the infinite period limit, as $m \rightarrow 1$, we have $ns(z) \rightarrow \coth z$ and $e_1 \rightarrow e_2$.

Therefore, using the fact that $e_1 + e_2 + e_3 = 0$, we get $e_3 = -2e_1$.

So, the corresponding EDMS is,

$$u(z) = \frac{-9\beta \pm 3\sqrt{9\beta^2 - 40\alpha\delta}}{\delta} [e_1 + 3e_1 \operatorname{cosech}^2 z]. \quad (3.10)$$

To derive the SWSs, we shall express the solutions in terms of reciprocal WEFs. So, we express the solution of Equation (3.4) in terms of reciprocal elliptic function by

$$u(z) = A + \frac{B}{\wp(z)}. \quad (3.11)$$

For convenience, we set the integration constant K in Equation (3.4) to zero. Then we substitute Equation (3.11) into the new Equation (3.4) with $K = 0$ and equate the coefficients of powers of $\frac{1}{\wp(z)}$ to zero, so that we arrive at $g_3 = 0$ and the following four equations

$$\frac{15}{2}\alpha g_2^2 - \frac{3}{2}\beta B g_2 + \frac{\delta}{3}B^2 = 0, \quad (3.12a)$$

$$\delta B = \frac{3}{2}\beta g_2, \quad (3.12b)$$

$$-18\alpha g_2 + 2\beta B + \delta A^2 = 0, \quad (3.12c)$$

$$-w + 2\beta B + \frac{\delta}{3} A^2 = 0. \quad (3.12d)$$

Solving for A , B and g_2 from Equations (3.12b), (3.12c) and (3.12d), we obtain

$$A = \pm \sqrt{\frac{w(18\alpha\delta - 3\beta)}{3\beta^2 + 6\alpha\delta - \beta}}, \quad (3.13a)$$

$$B = \frac{3w\beta}{2(3\beta^2 + 6\alpha\delta - \beta)}, \quad (3.13b)$$

$$g_2 = \frac{w\delta}{3\beta^2 + 6\alpha\delta - \beta}. \quad (3.13c)$$

Substituting for B in Equation (3.12a), we get the constraint relation for the coefficients α , β and δ as

$$5\alpha\delta = \beta^2. \quad (3.14)$$

Thus the doubly periodic wave solution for Equation (3.4) can be written as

$$u(z) = \pm \sqrt{\frac{w(18\alpha\delta - 3\beta)}{3\beta^2 + 6\alpha\delta - \beta}} + \frac{3w\beta}{2(3\beta^2 + 6\alpha\delta - \beta)} \frac{sn^2(z)}{(e_1 - e_3) + e_3 sn^2(z)}. \quad (3.15)$$

The corresponding SWS is,

$$u(z) = \pm \sqrt{\frac{w(18\alpha\delta - 3\beta)}{3\beta^2 + 6\alpha\delta - \beta}} + \frac{3w\beta}{2(3\beta^2 + 6\alpha\delta - \beta)} \frac{\tanh^2(z)}{e_1 + 2e_1 \operatorname{sech}^2(z)}. \quad (3.16)$$

4. Fifth Order mKdV Equation

Consider the fifth order mKdV equation

$$u_t + \beta u^2 u_x + c_3 u_{xxx} + c_5 u_{xxxxx} = 0, \quad (4.1)$$

where β , c_3 and c_5 are real constants.

TWSs of Equation (4.1) can be found assuming $z = x - wt$, where w is the wave velocity, so that Equation (4.1) becomes

$$-wu' + \beta u^2 u' + c_3 u''' + c_5 u^{(5)} = 0, \quad (4.2)$$

where ' denotes differentiation with respect to z .

Integrating Equation (4.2) with respect to z and setting the integration constant equal to zero, we obtain the equation

$$-wu + \frac{\beta}{3} u^3 + c_3 u'' + c_5 u^{(4)} = 0. \quad (4.3)$$

One can see like in the last section that Equation (4.3) has solution in the form of Equation (3.6) with A , B and g_2 given by

$$A = \mp \frac{6c_3}{\sqrt{-360c_5\beta}}, \quad (4.4a)$$

$$B = \pm \sqrt{-\frac{360c_5}{\beta}}, \quad (4.4b)$$

$$g_2 = -\frac{10c_5w + c_3^2}{180c_5^2}. \quad (4.4c)$$

The other invariant g_3 can be found using Equation (4.4a), (4.4b) and (4.4c) as

$$g_3 = \frac{2\beta A^3 - 6wA - 3Bg_2c_3}{12Bc_5}. \quad (4.4d)$$

From Equations (4.4a) and (4.4b), it is evident that β and c_5 should be of opposite signs. Also, since g_2 cannot be negative, the wave speed w should satisfy the inequality

$$w < -\frac{c_3^2}{10c_5}. \quad (4.5)$$

Thus, for positive speed, c_5 should be negative and for negative speed, c_5 should be positive.

So, the solution of Equation (4.3) is,

$$u(z) = \mp \frac{6c_3}{\sqrt{-360c_5\beta}} \pm \sqrt{-\frac{360c_5}{\beta}} \wp(z). \quad (4.6)$$

The solution in terms of Jacobi elliptic function can be written as

$$u(z) = \mp \frac{6c_3}{\sqrt{-360c_5\beta}} \pm \sqrt{-\frac{360c_5}{\beta}} [e_3 + (e_1 - e_3)ns^2(z)]. \quad (4.7)$$

As $m \rightarrow 1$, the solution (4.7) leads to the EDMS

$$u(z) = \mp \frac{6c_3}{\sqrt{-360c_5\beta}} \pm \sqrt{-\frac{360c_5}{\beta}} [e_1 + 3e_1 \operatorname{cosech}^2(z)]. \quad (4.8)$$

Now, to derive the SWSs we assume the solution of Equation (4.3) in terms of reciprocal WEFs as

$$u(z) = A + \frac{B}{\wp(z)}. \quad (4.9)$$

Substituting Equation (4.9) into Equation (4.3), we obtain

$$A = \pm \sqrt{-\frac{c_3^2}{10\beta c_5}}, \quad (4.10a)$$

$$B = \mp \frac{c_3^2 + 10c_5w}{120c_5^2} \sqrt{-\frac{10c_5}{\beta}}, \quad (4.10b)$$

$$g_2 = -\frac{c_3^2 + 10c_5w}{180c_5^2}, \quad (4.10c)$$

$$g_3 = 0. \quad (4.10d)$$

From Equation (4.10a), it is evident that for real solution, c_5 and β should be of opposite signs. For writing down Equation (4.10b), we assumed that c_3 and β are positive, so that c_5 must be negative. Also, since g_2 has to be always positive, we should have the condition that $c_3^2 + 10c_5w < 0$.

Thus our solution of Equation (4.3) in terms of JEF can be written as

$$u(z) = \pm \sqrt{-\frac{c_3^2}{10\beta c_5}} \mp \frac{c_3^2 + 10c_5 w}{120c_5^2} \sqrt{-\frac{10c_5}{\beta}} \left[\frac{sn^2(z)}{(e_1 - e_3) + e_3 sn^2(z)} \right]. \quad (4.11)$$

The corresponding SWS is given by

$$u(z) = \pm \sqrt{-\frac{c_3^2}{10\beta c_5}} \mp \frac{c_3^2 + 10c_5 w}{120c_5^2} \sqrt{-\frac{10c_5}{\beta}} \left[\frac{\tanh^2(z)}{e_1 + 2e_1 \operatorname{sech}^2(z)} \right]. \quad (4.12)$$

5. Coupled Schrodinger-KdV System

The coupled Schrodinger-KdV system

$$\begin{aligned} iu_t &= u_{xx} + uv, \\ v_t + 6vv_x + v_{xx} &= (|u|^2)_x \end{aligned} \quad (5.1)$$

is known to describe various processes in dusty plasma, such as Langmuir, dust acoustic wave and electro-magnetic waves. The complete integrability of the system has been studied by Chowdhury et al. [3]. A kind of solution was obtained by Hase and Satsuma [8].

We introduce the transformations

$$\begin{aligned} u &= e^{i\theta} U(z), \quad v = V(z), \\ \theta &= px + qt, \quad z = kx + ct, \end{aligned} \quad (5.2)$$

where p, q, k and c are constants.

Substituting Equation (5.2) into Equation (5.1), we find that $c = 2pk$, and U, V satisfy the following coupled ordinary differential system

$$\begin{aligned} k^2 U'' + (q - p^2) U + UV &= 0, \\ 2pkV' + 6kVV' + k^3V''' - k(U^2)' &= 0. \end{aligned} \quad (5.3)$$

We assume a solution of Equation (5.3) in the form

$$\begin{aligned} U &= A + B\wp(z), \\ V &= C + D\wp(z). \end{aligned} \quad (5.4)$$

Substituting Equation (5.4) into Equation (5.3), we obtain

$$A = \pm\sqrt{2}(q - p^2), \quad (5.5a)$$

$$B = \pm 6\sqrt{2}k^2, \quad (5.5b)$$

$$C = 0, \quad (5.5c)$$

$$D = -6k^2, \quad (5.5d)$$

$$g_2 = \frac{(q - p^2)^2}{3k^4}. \quad (5.5e)$$

We can see that g_3 is arbitrary and the constraint relation among the coefficients is,

$$2p^2 - p - 2q = 0. \quad (5.6)$$

Thus the doubly PWSs for Equation (5.3) are,

$$U(z) = \pm\sqrt{2}(q - p^2) \pm 6\sqrt{2}k^2[e_3 + (e_1 - e_3)ns^2(z)], \quad (5.7a)$$

$$V(z) = -6k^2[e_3 + (e_1 - e_3)ns^2(z)]. \quad (5.7b)$$

In the infinite period limit, as $m \rightarrow 1$, the EDMSs are

$$U(z) = \pm\sqrt{2}(q - p^2) \pm 6\sqrt{2}k^2[e_1 + 3e_1 \operatorname{cosech}^2(z)], \quad (5.8a)$$

$$V(z) = -6k^2[e_1 + 3e_1 \operatorname{cosech}^2(z)]. \quad (5.8b)$$

6. Conclusion

The generalized fifth order KdV equation has been found to have EDMSs as well as SWSs by the method of WEF with no restrictions on the signs of the coefficients. It has been found that the fifth order mKdV equation has the EDMSs and the SWSs with the coefficients of the nonlinear term and the third order dispersive term being positive and the coefficient of the fifth order dispersive term being negative. The coupled nonlinear Schrodinger-KdV system has also been found to have EDMSs with a simple constraint relation among some of the coefficients.

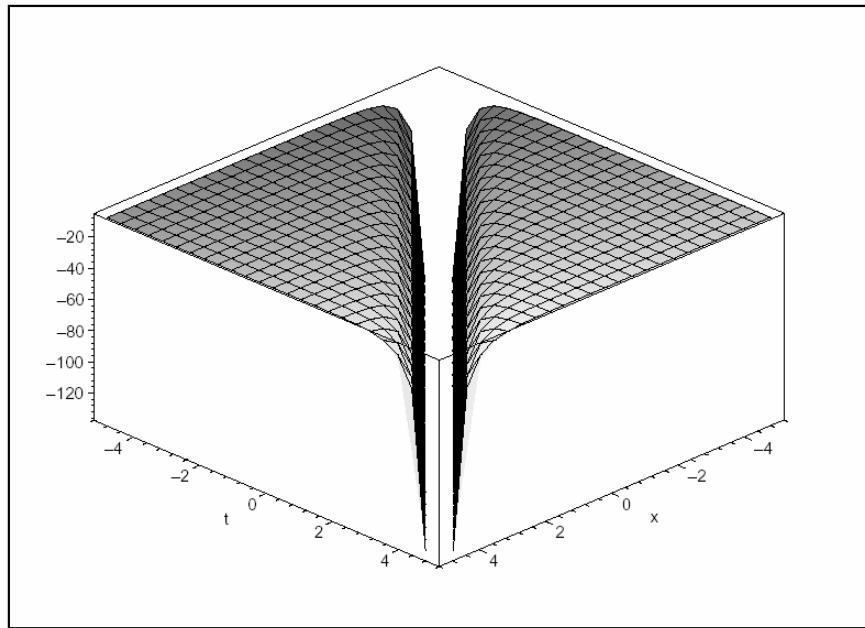


Figure 1. EDMS (3.10) with $\alpha=1$, $\beta=3$, $\delta=1$, $\omega=1$, $e_1=1$.

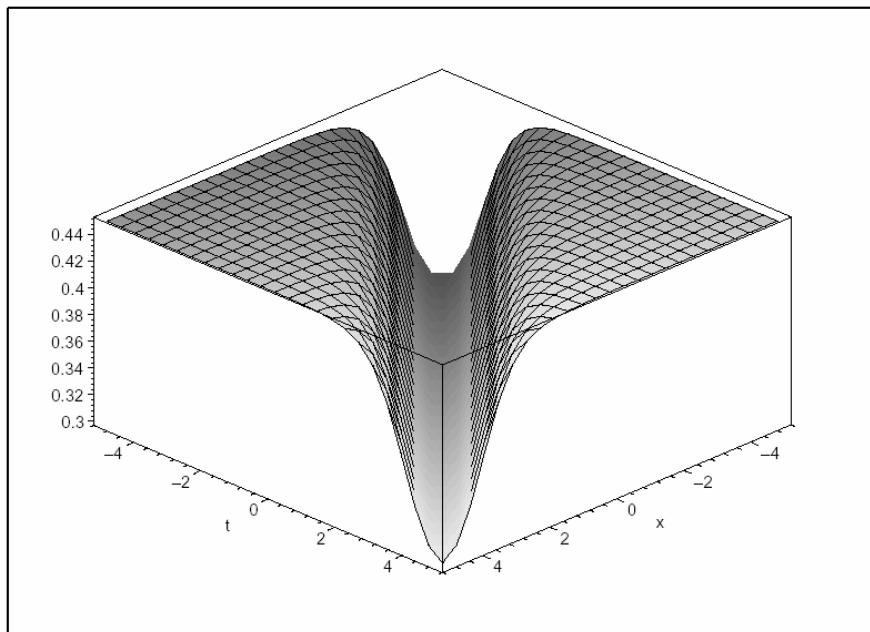


Figure 2. SWS (3.16) with $\alpha=1$, $\beta=3$, $\delta=1$, $\omega=1$, $e_1=1$.

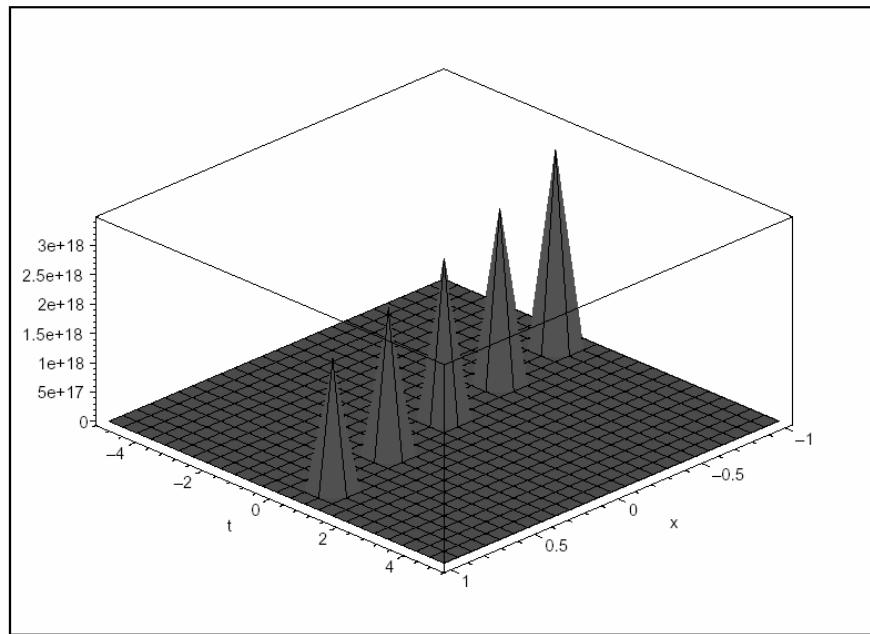


Figure 3. EDMS (4.8) with beta=3, omega=1, e1=1, c3=1, c5=-1.

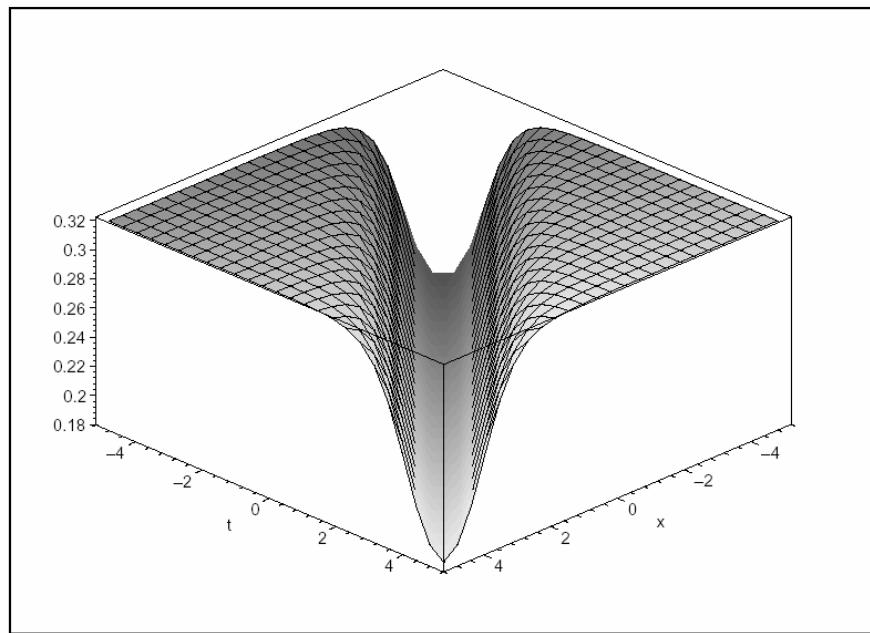


Figure 4. SWS (4.12) with beta=3, omega=1, e1=1, c3=1, c5=-1.

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